Complex Angular Momentum in Three-Particle Potential Scattering*

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In this paper we discuss the continuation of a class of three-particle potential scattering amplitudes to complex values of the total angular momentum. The class of amplitudes considered are those which describe a scattering in which a given pair of the particles is bound in the initial and final states. A nonrigorous discussion indicates that, except for simple kinematic factors, the only singularities present in the complex angular momentum plane are dynamical poles and possible isolated essential singularities. The Watson-Sommerfeld transformation of the full amplitude can be performed to display its large momentum-transfer behavior.

I. INTRODUCTION

THE behavior of two-particle potential scattering amplitudes at large momentum transfers has been successfully examined in terms of poles in the complex angular momentum plane. In the previous paper (denoted here by I) we have discussed some general features involved in determining the large momentum-transfer behavior of many-particle potential scattering amplitudes by complex angular momentum techniques. In particular, a continuation of the partial-wave Schrödinger equation to complex values of the angular momentum was obtained. A nonrigorous investigation of the analytic properties of a class of three-particle amplitudes found from the scattering solutions to this equation is given in this paper. With an appropriate choice of boundary conditions for the scattering solution, we find that this continuation determines the large momentum-transfer behavior of the full amplitude. Moreover, the singularities in the angular momentum plane which determine the asymptotic behavior consist only of poles and possibly isolated essential singularities. The simplicity of Regge's description of the asymptotic behavior of scattering amplitudes is thus maintained in this three-particle case.

Since any multiparticle system has only three degrees of rotational freedom, all of the angular momentum features of the many-particle problem are already present in a system of three particles. We will therefore be concerned in the bulk of this paper with three-particle scattering and later indicate how the generalization to nonrelativistic many-particle amplitudes can be made. For simplicity, we are considering spinless and nonidentical particles interacting by means of two-body Yukawa potentials.

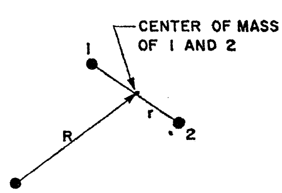

FIG. 1. Position vectors for a three-particle system in the center-of-mass frame.

In a three-particle scattering process, there are several amplitudes that can be discussed depending on how much of the interaction is turned off in the asymptotic states. In the initial and final states one can have either three free particles or a bound pair with the third free. Each possible amplitude represents a different class of scattering boundary conditions.[1] To avoid discussing these various amplitudes at length, we will concentrate on the particular class of amplitudes in which initially and finally particles 1 and 2 are bound. This particular class is at once interesting from the point of view of real processes and simple to calculate.

To determine the analytic properties of these amplitudes we will exploit the kinematical similarity that three-particle scattering bears to the scattering of two particles with spin. If initially and finally two of the three particles are in a definite state of their relative angular momentum l, then the scattering is kinematically the same as the scattering of a particle from a composite object with a spin l and certain other internal degrees of freedom. Of course, this spin is not conserved and there is a continuous infinity of internal degrees of freedom corresponding to the energy of the composite object.

II. THE PARTIAL-WAVE SCHRÖDINGER EQUATION

In order to study the solutions of Schrödinger's equation, we will introduce a specific coordinate system and make explicit the procedures outlined in I. We begin by surpressing the three degrees of freedom corresponding to the total center of mass. The wave function then depends on two position vectors which we may take to be $\mathbf{r}$, the relative coordinate of particles 1 and 2; and $\mathbf{R}$, the coordinate of their center of mass relative to the third particle (see Fig. 1). These coordinates were denoted collectively by $\mathbf{r}$ in I. The scattering wave function also depends on the quantum numbers which label the incoming wave, denoted by $\mathbf{p}$ in I. These will be chosen to be $\mathbf{P}$, the total momentum of the composite object 1 and 2 and the quantum numbers which characterize its internal wave function.

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[1] For a summary of some of these aspects of three-particle scattering amplitudes, see C. Lovelace, Three-Particle Systems and Unstable Particles, Lectures at Edinburgh Summer School, 1963 (to be published), and the references cited therein.

The latter labels will be taken to be l, the relative angular momentum of particles 1 and 2, their "helicity" $\eta=\mathbf{l}\cdot\mathbf{P}/|\mathbf{P}|$, and their center-of-mass energy, $p^2/2m$. Since $\mathbf{P}\cdot(\mathbf{L}-\mathbf{l})=\mathbf{P}\cdot(\mathbf{R}\times\mathbf{P})=0$, the helicity is also the projection of $\mathbf{L}$ on $\mathbf{P}$. The total conserved energy is

$$E=(p^2/2m)+(\mathbf{P}^2/2m'). \tag{2.1}$$

Here, if m_1, m_2, m_3 are the particle masses, we have

$$m=\frac{m_1m_2}{m_1+m_2},\quad m'=\frac{(m_1+m_2)m_3}{m_1+m_2+m_3}. \tag{2.2}$$

Explicitly, the wave function is written:

$$\Psi=\Psi[(lp\eta)\mathbf{P};\,\mathbf{r},\mathbf{R}].$$

This choice of variables is clearly suitable for pursuing the analogy with the scattering of particles with spin.

Now introduce polar coordinates R, θ, φ of $\mathbf{R}$ relative to some arbitrary polar axis, and r, β, α of $\mathbf{r}$ defined with $\mathbf{R}$ as a polar axis. It is also convenient to introduce a fixed and arbitrary angle α' to define the origin of α (see Fig. 2).

The three angles φ, θ, $\psi=\alpha+\alpha'$ are the Euler angles discussed in I. The corresponding body-fixed z axis lies along $\mathbf{R}$. The potential, which depends only on the interparticle distances, is a function of the remaining coordinates R, r, β.

We will now write down the Schrödinger equation in these coordinates and make the partial-wave expansion of I. The full Schrödinger equation for the wave function $\Phi(\mathbf{r},\mathbf{R})=\Psi(\mathbf{r},\mathbf{R})/Rr$, is (with $\hbar=1$)

$$\left[\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2}-\frac{\mathbf{l}^2}{r^2}\right)+\frac{1}{2m'}\left(\frac{\partial^2}{\partial R^2}-\frac{(\mathbf{L}-\mathbf{l})^2}{R^2}\right)+E-V(R,r,\beta)\right]\times\Phi(\mathbf{r},\mathbf{R})=0. \tag{2.3}$$

Here, $\mathbf{L}$ and $\mathbf{l}$ are regarded as differential operators in the angles.

The wave function can be expanded in a complete set of functions for the variables r, β, α, φ, θ obtaining a set of coupled differential equations in the coordinate R. To do this, we make the partial-wave expansion of I taking $\mathbf{P}$ to be the space-fixed z axis. We will thus be expanding in eigenfunctions labeled by L, its projection on the space-fixed axis η, and the projection on the body-fixed axis η'

$$\eta'=\mathbf{L}\cdot\mathbf{R}/|\mathbf{R}|. \tag{2.4}$$

We also expand in the eigenfunctions of the relative angular momentum l and its projection on $\mathbf{R}$. Since $\mathbf{L}\cdot\mathbf{R}=\mathbf{l}\cdot\mathbf{R}$, the combined orthogonal eigenfunctions can be written

$$((2L+1)/4\pi)D_{\eta\eta'}{}^L(\varphi,\theta,\alpha')Y_l{}^{\eta'}(\beta,\alpha). \tag{2.5}$$

Finally, we will expand in a complete set of solutions for the variables r. These will be the solutions $\psi_l(p,r)$ of the two-particle problem

$$\left[\frac{d^2}{dr^2}+p^2-\frac{l(l+1)}{r^2}-2mV(r)\right]\psi_l(p,r)=0. \tag{2.6}$$

The complete expansion then becomes

$$\Phi[(lp\eta)\mathbf{P};\,\mathbf{r},\mathbf{R}]=(4\pi)^{-1}\sum_L\sum_{l'p'\eta'}(2L+1)\phi_{l'p'\eta',lp\eta}{}^L(R)\psi_{l'}(p',r)\times D_{\eta\eta'}{}^L(\varphi,\theta,\alpha')Y_{l'}{}^{\eta'}(\beta,\alpha). \tag{2.7}$$

The sum is over the discrete and continuous spectrum of the two-particle problem in p ($\sum_p$ denotes an integral), over integral values of l with $l\geq|\eta'|$, and over η' with $|\eta'|\leq L$.

In proejcting out the equations which govern the $\phi_{l'p'\eta',lp\eta}{}^L(R)$, only the matrix elements of $\mathbf{l}\cdot\mathbf{L}$ and $V(R,r,\beta)$ are not readily evaluated. In order to evaluate the former, it is convenient when considering rotations generated by $\mathbf{L}$ to maintain α fixed and let α' vary, while when considering rotations generated by $\mathbf{l}$, we keep α' fixed and let α vary. In terms of the spherical components[2] of $\mathbf{L}$ and $\mathbf{l}$ in the body-fixed frame

$$\mathbf{L}(\varphi,\theta,\alpha')\cdot\mathbf{l}(\beta,\alpha)=-L_+l_--L_-l_++L_0l_0. \tag{2.8}$$

The matrix elements of Eq. (2.8) are then given by Eq. (A1) of I recalling that $\mathbf{l}$ is an angular momentum like $\mathbf{L}$, and $Y_l{}^\eta(\beta,\alpha)=D_{\eta 0}{}^L(\beta,\alpha,0)$.

Define

$$a_\eta=[(L-\eta)(L+\eta+1)(l-\eta)(l+\eta+1)]^{1/2}, \tag{2.9}$$

$$V_{lp\eta,l'p'\eta'}=2m'\delta_{\eta\eta'}\int d^3r\psi_{l'}{}^*(p',r)Y_{l'}{}^{\eta'*}(\beta,\alpha)[V(R,r,\beta)-V(r)]\times\psi_l(p,r)Y_l{}^\eta(\beta,\alpha)/r^2, \tag{2.10}$$

where $V(r)$ is the interaction between particles 1 and 2. The equation becomes[3]:

$$\left[\frac{d^2}{dR^2}+P'^2-\frac{L(L+1)+l(l+1)-2\eta'^2}{R^2}\right]\phi_{l'p'\eta',lp\eta}{}^L(R)+(1/R^2)[a_{\eta'}\phi_{l'p'\eta'+1,lp\eta}{}^L(R)+a_{-\eta'}\phi_{l'p'\eta'-1,lp\eta}{}^L]-\sum_{l''p''\eta''}V_{l'p'\eta',l''p''\eta''}(R)\phi_{l''p''\eta'',lp\eta}{}^L(R)=0. \tag{2.11}$$

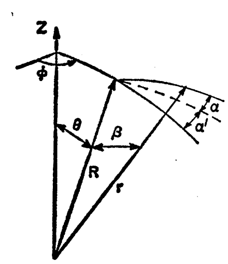

FIG. 2. Polar coordinates for the vectors $\mathbf{r}$, $\mathbf{R}$. Z is a space-fixed axis.

[2] A. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[3] The coordinate transformation can also be performed explicitly; see E. Hylleraas, Z. Physik **48**, 469 (1928).

This can be written in a matrix form as

$$\left[\frac{d^2}{dR^2}+P^2-\frac{\Lambda^L}{R^2}-V(R)\right]\phi^L(R)=0. \qquad (2.12)$$

The intermediate η'' sum remains restricted to $|\eta''|\leq L$.

This equation displays the formal equivalence of the three-particle problem to a problem having many two-body channels. In Eq. (2.7) we can regard $\psi_l(p,r)Y_l^\eta(\beta,\alpha)$ as the internal wave function of a composite particle composed of particles 1 and 2 which scatters off particle 3. The equation which governs such a set of two-body channels is (2.12).

III. THE EQUATION AND SOLUTIONS FOR COMPLEX ANGULAR MOMENTUM

We continue the equations to complex L as discussed in I. The restriction that $|\eta'|\leq L$ is removed and the coupled equations are considered for arbitrary integral values of η'. When L is an integer, a_L vanishes thus guaranteeing that the physical equations will decouple from the unphysical ones.

In order to examine the analytic properties of the solutions, it is convenient to introduce a wave function $\hat{\phi}^L$ which obeys an equation all of whose coefficients are entire functions of L.

$$\phi^L=\rho\hat{\phi}^L\rho\,,$$

where[4]

$$\rho_{lp\eta,l'p'\eta'}=\delta_{ll'}\delta_{pp'}\delta_{\eta\eta'}[(L-\eta)!(L+\eta)!]^{-1/2}. \qquad (3.1)$$

The potential, a function of the interparticle distances, is independent of the angle α and hence its matrix elements are diagonal in η [see Eq. (2.10)]. The equation governing $\hat{\phi}^L$ can then be written

$$\left[\frac{d^2}{dR^2}+P^2-\frac{\hat{\Lambda}^L}{R^2}-V(R)\right]\hat{\phi}^L(R)=0. \qquad (3.2)$$

Here, $\hat{\Lambda}^L$ is defined in the same way as Λ^L with a_η replaced by

$$\hat{a}_\eta=(L-\eta)[(l-\eta)(l+\eta+1)]^{1/2}. \qquad (3.3)$$

Every element of $\hat{\Lambda}^L$, and therefore every element of the matrix of equations, is an entire function of L.

We will now examine the analytic properties of the solutions to Eq. (3.2) for two simple classes of boundary conditions. From these solutions, the solution to the scattering problem can be constructed and the analytic properties of the amplitude determined.

In the succeeding paragraphs, many questions of convergence will of necessity be left unanswered. We will treat the coupled set of equations (3.2) as though it were a finite matrix of equations. This can be done by introducing a cutoff in the intermediate p and l sums. The true S matrix will be the limit as the cutoff tends to infinity of the S matrices computed from these truncated equations. The analytic properties we derive are those of each term in the sequence. The potential which couples the equations together is independent of the angular momentum, so it is perhaps plausible that the limit has the same analytic properties as each term in the sequence. We are well aware that mathematically this may not be the case and we have been unable to find a rigorous proof for the statement.

Certain intermediate steps in the proof, such as the convergence of the several series used to define the solutions, will also not be discussed in this paper. Some of them, however, have been proved rigorously and we will indicate in a later paragraph which these are.

We first obtain solutions of Eq. (3.2) specified by boundary conditions at the origin. In order to examine their analytic properties, we employ the standard power-series technique[5,6] and look for solutions of the form

$$\hat{\phi}^L(R)=\Big[\sum_{n=0}^{\infty}a(n,\sigma)R^n\Big]R^\sigma. \qquad (3.4)$$

Here, σ and a are matrices, and R^σ is a matrix whose diagonal elements have the form R^{σ_ξ} when σ is diagonal and has eigenvalues σ_ξ.

Near the origin, the matrix elements of a Yukawa potential increase no faster than R^{-1}, so that

$$a(0,\sigma)[\sigma(\sigma-1)]-\hat{\Lambda}^La(0,\sigma)=0. \qquad (3.5)$$

We will demand that σ, $a(0,\sigma)$, and $\hat{\Lambda}^L$ be simultaneously diagonal. $\hat{\Lambda}^L$ is the orbital angular momentum $(\mathbf{L}-\mathbf{l})$ of particles 1 and 2. It is diagonal in l. The eigenvalues of a given submatrix characterized by l are discussed in Appendix A and given by

$$(L+\xi)(L+\xi+1) \quad \xi \text{ integral}, \quad |\xi|\leq l \qquad (3.6)$$

including the case of complex L.

The allowed values of σ_ξ are thus from Eq. (3.5)

$$\sigma_\xi'=L+\xi+1\,, \quad \sigma_\xi''=-L-\xi. \qquad (3.7)$$

The transformation which diagonalizes $\hat{\Lambda}^L$ is discussed in Appendix A and is denoted by

$$\hat{U}_{lp\eta,l'p'\xi}=\delta_{ll'}\delta_{pp'}\hat{U}_{\eta\xi}. \qquad (3.8)$$

The index ξ refers to the representation in which $\hat{\Lambda}^L$ is diagonal. In this representation it is demonstrated in Appendix A that for small R the potential matrix elements have the behavior

$$V_{lp\xi,l'p'\xi'}\propto R^{|\xi-\xi'|}. \qquad (3.9)$$

Recalling that $a_{lp\xi,l'p'\xi'}(0,\sigma)\propto\delta_{\xi\xi'}$, an inspection of the differential equation shows that the leading terms in the series solution of Eq. (3.4) then have the form $\hat{\phi}_{lp\xi,l'p'\xi'}{}^L\propto R^{\sigma_\xi}$ rather than $\propto R^{\sigma_{\xi'}}$. Equation (3.4) can thus be

[4] For complex z, we define $z!=\Gamma(z+1)$. $\delta_{pp'}$ is to be interpreted as $\delta(p-p')$.

[5] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962).

[6] J. Charap and E. Squires, Ann. Phys. (N. Y.) **20**, 145 (1962); R. Newton, Phys. Rev. **129**, 1437 (1963).

written

$$\hat{\phi}^L(R)=R^\sigma\sum_{n=0}^{\infty}a(n,\sigma)R^n. \qquad (3.10)$$

If we make the expansion

$$V(R)-P^2=\sum_{m=0}^{\infty}b(m)R^m, \qquad (3.11)$$

the Schrödinger equation determines the recursion relation

$$[(\sigma+n+2)(\sigma+n+1)-\hat{\Lambda}^L]a(n+2,\sigma)=\sum_{k=0}^{n}b(n-k)a(k,\sigma). \qquad (3.12)$$

The fact that $\hat{\phi}^L(R)$ can be written in the form of Eq. (3.10) is useful because the complications usually associated with indices σ_ξ which differ by integers are absent in this case.[7] Except for certain integer or half-integer values of L, $[(\sigma+n)(\sigma+n-1)-\hat{\Lambda}^L]^{-1}$ is regular for all n.

For the set of indices σ_ξ', it is convenient to choose

$$a_{lp\eta,l'p'\eta'}(0,\sigma')=\delta_{ll'}\delta_{pp'}\pi^{1/2}\sum_\xi \hat{U}_{\eta\xi}(\tfrac{1}{2}PR)^{\sigma_\xi'}\times e^{-i\pi[\frac{1}{2}(\xi-L)-l]}[(\sigma_\xi'-\tfrac{1}{2})!]^{-1}\hat{U}_{\xi\eta'}^{-1}. \qquad (3.13)$$

With this choice of boundary condition the solution of the free Schrödinger equation will be $j(R)$ defined in Appendix B. The products $\hat{U}_{\eta\xi}\hat{U}_{\xi\eta'}^{-1}$ are meromorphic functions of L (see Appendix A). From Eq. (3.11) we can conclude that $a(n,\sigma')$ are also meromorphic with additional poles due to those of $[(\sigma'+n)(\sigma'+n-1)-\hat{\Lambda}^L]^{-1}$. These poles are fixed and kinematic and do not depend on R. Formally, the same analytic properties hold for the sum, Eq. (3.9). This solution will be denoted by $J(R)$. Using the power-series method this solution may be extended to all real values of R.

Similar analytic properties hold for the solutions with indices σ_ξ'' for which we choose

$$a_{lp\eta,l'p'\eta'}(0,\sigma'')=-\delta_{ll'}\delta_{pp'}\pi^{-1/2}\sum_\xi \hat{U}_{\eta\xi}2P^{-1}\times(\tfrac{1}{2}PR)^{\sigma_\xi''}e^{i\pi[\frac{1}{2}(\xi-L)-l]}(-\sigma_\xi''-\tfrac{1}{2})!\hat{U}_{\xi\eta'}^{-1}. \qquad (3.14)$$

This solution is denoted by $N(R)$.

The general solution of the Schrödinger equation may be expressed as a matrix linear combination of $J(R)$ and $N(R)$. For physical values of L, $J(R)$ is the solution regular near $R=0$. We will, therefore, require that the scattering solution be a matrix multiple of $J(R)$ as a boundary condition for complex L. $J(R)$ contains irregular solutions for general complex L.

The scattering solution consists of the free solution plus a scattered part which contains only outgoing or decaying waves. As a second class of boundary conditions, we will consider elementary solutions of this type. Solutions having outgoing waves in the open channels ($p^2/2m<E$) and decaying waves in the closed ones ($p^2/2m>E$) are generated by the free Green's function:

$$G^L(R,R')=j(R_<)h^{(1)}(R_>). \qquad (3.15)$$

Here, $j(R)$ is the free solution regular at the origin for physical L, and $h^{(1)}(R)$ is a free solution corresponding to pure outgoing waves in the open channels or decaying waves in the closed channels. For the closed channels in which the energy of the composite object is greater than that of the total system, P is defined with positive imaginary part. These solutions are given explicitly in Appendix B. There it is shown that both $j(R)$ and $h^{(1)}(R)$ are entire functions of L so that the Green's function is also.

The free solution corresponding to $h^{(1)}(R)$ but defined with incoming or expanding waves is denoted by $h^{(2)}(R)$ and discussed in Appendix B. We introduce the free solution $h^{(3)}(R)$ defined by

$$h_{lp\eta,l'p'\eta'}^{(3)}(R)=\delta_{pp'}h_{lp\eta,l'p'\eta'}^{(2)}(R) \quad p^2/2m<E$$
$$=0 \quad p^2/2m>E. \qquad (3.16)$$

$h^{(3)}(R)$ is an entire function of L containing only incoming and no expanding waves. Solutions of the complete Schrödinger equation (3.2) are then defined by

$$H^{(1,3)}(R)=h^{(1,3)}(R)+\int_{R_0}^{\infty}dR'j(R_<)h^{(1)}(R_>)\times V(R')H^{(1,3)}(R') \quad R\geq R_0>0. \qquad (3.17)$$

The Fredholm method[8] may be used to construct solutions to these equations. One obtains

$$H^{(1,3)}(R)=h^{(1,3)}(R)+\frac{\Delta^{(1,3)}(R)}{D(E,L)}. \qquad (3.18)$$

Here, $\Delta^{(1,3)}(R)$ is a matrix and $D(E,L)$ a function both expressed as Fredholm series on the kernel of Eq. (3.17). For our present purpose, it suffices to recall the form of these series. The nth term in the series for $D(E,L)$ consists of an integral over a determinant of an $n\times n$ matrix whose entries are the kernel of Eq. (3.17). A similar statement holds for $\Delta^{(1,3)}(R)$ except that there is an additional integration over $h^{(1,3)}(R)$. Since the kernel is an entire function of L, and $h^{(1,3)}(R)$ is also, both nth order terms are entire functions of L. With the qualifications on rigor mentioned above, we can then conclude that both $D(E,L)$ and every element of $\Delta^{(1,3)}(R)$ are entire functions of L. The solutions $H^{(1,3)}$ are thus analytic in the complex angular momentum plane except at dynamical poles arising from the zeros of $D(E,L)$.

The Green's function, Eq. (3.15), is a diagonal matrix in the indices l and p. A given diagonal element $G_{lp\eta,lp'\eta'}{}^L(R,R')$ will be irregular for small values of

[7] Compare E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill Book Company, Inc., New York, 1955), Chap. 4.

[8] A. C. Zaanen, *Linear Analysis* (P. Noordhoff, Gronigen, 1953).

$R_<$ if $\text{Re}L<l-1$ as can be seen from the explicit expression for $j(R)$ in Appendix B. Since arbitrarily large values of l are coupled by Eq. (3.17), there is no value of L for which every element of the kernel of Eq. (3.17) is regular for small $R_<$. The Fredholm method thus cannot be straightforwardly applied unless R_0 is greater than zero.

Every solution of Schrödinger's equation cannot be expressed as a matrix linear combination of the two solutions $H^{(1,3)}(R)$ since in Eq. (3.16) we have omitted the expanding waves. The scattering solution, however, contains no expanding waves and can be expressed as a linear combination of these solutions for $R>R_0$.

IV. THE SCATTERING AMPLITUDE

The asymptotic states which contain particles 1 and 2 bound and particle 3 free can be characterized by the quantum numbers $\mathbf{P}(lp\eta)$. The scattering amplitude will therefore be denoted by $\langle(l'p'\eta')\mathbf{P}'|T|\mathbf{P}(lp\eta)\rangle$, and is a function of the angles $\cos\theta=\mathbf{P}\cdot\mathbf{P}'/|\mathbf{P}||\mathbf{P}'|$ and φ an azimuthal angle of $\mathbf{P}'$ about $\mathbf{P}$. Since there are only two vectors which characterize the amplitude, the angle ψ of the partial-wave expansion discussed in I is superfluous and may be chosen as $-\varphi$. The partial-wave expansion is then[9]

$$\langle(l'p'\eta')\mathbf{P}'|T|\mathbf{P}(lp\eta)\rangle = (4\pi)^{-1}\sum_L(2L+1)T_{l'p'\eta',lp\eta}{}^L(E) \times D_{\eta\eta'}{}^L(\varphi,\theta,-\varphi). \quad (4.1)$$

The sum ranges are integral values of $L\geq\max(|\eta|,|\eta'|)$.

At large R, the scattering[3] solution is a sum of two parts. First, there is the product of incoming plane wave and a bound-state wave function characterized by $\eta=\mathbf{l}\cdot\mathbf{P}/|\mathbf{P}|$. The second part consists of outgoing spherical waves times states of definite $l'p'$ and $\eta'=\mathbf{l}\cdot\mathbf{R}/|\mathbf{R}|$.

$$\Psi[(lp\eta)\mathbf{P};\mathbf{r},\mathbf{R}]\to e^{i\mathbf{P}\cdot\mathbf{R}}\psi_l(p,r)\sum_{\eta'}D_{\eta\eta'}{}^l(\varphi,\theta,-\varphi) \times Y_l{}^{\eta'}(\beta,\alpha)+\sum_{l'p'\eta'}(e^{iP'R}/R)f_{l'p'\eta',lp\eta}(\mathbf{P}',\mathbf{P}) \times\psi_{l'}(p',r)Y_{l'}{}^{\eta'}(\beta,\alpha), \quad (4.2)$$

where

$$\frac{\mathbf{P}'}{|\mathbf{P}'|}=\frac{\mathbf{R}}{|\mathbf{R}|}\quad E=\frac{P'^2}{2m'}+\frac{p^2}{2m}. \quad (4.3)$$

The rotation matrices in the incoming part result from expressing states of definite $\mathbf{l}\cdot\mathbf{P}/|\mathbf{P}|$ as superpositions of states of definite $\mathbf{l}\cdot\mathbf{R}/|\mathbf{R}|$. Asymptotically, final states of definite $\mathbf{l}\cdot\mathbf{R}/|\mathbf{R}|$ become states of definite helicity $\mathbf{l}\cdot\mathbf{P}'/|\mathbf{P}'|$. The sum over final states is taken over all accessible $\psi_{l'}(p',r)$ including those in the continuum. In the case where p and p' correspond to bound states, we may take this expression to define the scattering amplitude

$$\langle(l'p'\eta')\mathbf{P}'|T|\mathbf{P}(lp\eta)\rangle = (1/2\pi)(P'P)^{1/2}f_{l'p'\eta',lp\eta}(\mathbf{P}',\mathbf{P}). \quad (4.4)$$

These expressions resemble those of the scattering of particles with spin where the $\psi_l(p,r)Y_l{}^\eta(\beta,\alpha)$ play the part of internal wave functions.[10]

The asymptotic behavior of $\hat{\phi}^L(R)$ may be projected out of Eq. (4.2). This involves replacing the plane wave by its familiar Rayleigh expansion evaluated at large R and performing some Clebsch-Gordan sums arising from the angular integration. Apart from a constant, one has

$$\phi_{l'p'\eta',lp\eta}{}^L(R)\underset{R\to\infty}{\longrightarrow}\delta_{pp'}\delta_{-\eta\eta'}\delta_{ll'}e^{-iPR}(P)^{-1/2} - e^{-i\pi(L+l)}e^{iP'R}(P')^{-1/2}S_{l'p'\eta',lp\eta}{}^L(E), \quad (4.5)$$

where we have introduced the S matrix

$$S=I+iT. \quad (4.6)$$

Equation (4.5) is taken as the definition of the partial-wave amplitude for complex L.

The amplitude thus defined can be found by constructing the scattering solution from the elementary solutions discussed earlier. To do this we use a method which differs slightly from that employed by Regge and others for the case of two-body channels.[6] That approach would involve the knowledge of solutions which behave like expanding waves in the closed channels and these we have not computed.

The scattering solution Eq. (4.5) has a unit amount of incoming wave. Noting the asymptotic properties of $H^{(3)}(R)$ implicit in Eq. (B4), it can be written as a superposition of $H^{(1)}(R)$ and $H^{(3)}(R)$ in the form

$$\hat{\phi}^L(R)=H^{(3)}(R)+H^{(1)}(R)X(E,L). \quad (4.7)$$

The matrix X is to be determined by the boundary condition at the origin. Since $J(R)$ and $N(R)$ are also a complete set of solutions to the Schrödinger equation, we may express $H^{(1,3)}(R)$ in terms of them.

$$H^{(1,3)}(R)=J(R)A^{(1,3)}+N(R)B^{(1,3)}. \quad (4.8)$$

In order to determine the constants, we may introduce an analog of the Wronskian in a many-channel problem. If χ and ψ are solutions of the Schrödinger equation, Eq. (3.2), then

$$W[\chi,\psi]=\chi^T\rho^2\frac{\partial\psi}{\partial R}-\frac{\partial\chi^T}{\partial R}\rho^2\psi \quad (4.9)$$

is independent of R. This is a simple consequence of the symmetry of the potential and Λ^L. The boundary conditions on the function $J(R)$, $N(R)$ imply

$$W[J,N]=I,\quad W[J,J]=W[N,N]=0. \quad (4.10)$$

In the region where the series for J and N converge, we may evaluate

$$A^{(1,3)}=-W[N,H^{(1,3)}],\quad B^{(1,3)}=W[J,H^{(1,3)}]. \quad (4.11)$$

[9] Compare M. Jacob and G. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

[10] R. Newton, J. Math. Phys. 1, 319 (1960).

The scattering solution can then be written

$$\hat{\phi}^L(R)=J(R)[A^{(1)}X+A^{(3)}] \\ +N(R)[B^{(1)}X+B^{(3)}]. \quad (4.12)$$

The boundary condition at the origin is that the scattering solution be a multiple of $J(R)$. This condition determines the matrix X.

$$X=-[B^{(1)}]^{-1}B^{(3)}. \quad (4.13)$$

The S matrix may be expressed in terms of X by taking the asymptotic limit of Eq. (4.7) and comparing with Eq. (4.5). This can be done by consulting Appendix B and defining

$$Z^{(1,3)}=\int_{R_0}^{\infty} dRj(R)V(R)H^{(1,3)}(R). \quad (4.14)$$

Putting $S=\rho\hat{S}\rho^{-1}$, we have for $\hat{S}$

$$\hat{S}=[I+Z^{(1)}]X+Z^{(3)}. \quad (4.15)$$

This formula holds only for those elements $\hat{S}_{lp\eta,l'p'\eta'}$ such that p and p' correspond to open channels.

The analytic properties of $\hat{S}$ can now be read off Eq. (4.15) knowing those of X and $Z^{(1,3)}$. From the properties of j and $H^{(1,3)}$ discussed in the previous section, we have that $Z^{(1,3)}$ are meromorphic functions of L with only the dynamical poles of $H^{(1,3)}$ as singularities.

From Eqs. (3.17), (3.13), (3.12), it can be seen that the solutions $J(R)$ and $H^{(1)}(R)$ have the form

$$J(R)=j(R)+(\text{terms depending on potential}),$$
$$H^{(1)}(R)=h^{(1)}(R)+(\text{terms depending} \\ \text{on potential}). \quad (4.16)$$

The free solutions are diagonal in p while the remaining matrix elements have a smooth dependence on p and p'. From Eq. (4.11), $B^{(1)}$ has the form

$$B^{(1)}=I+M, \quad (4.17)$$

where the matrix M has a smooth dependence on p and p'. In this form, the Fredholm theory can be applied to invert $B^{(1)}$.

The matrices $B^{(1)}$ and $B^{(3)}$ will have the isolated kinematic singularities at integer and half-integer values of L possessed by the solution J. We will assume that these kinematic singularities do not determine the asymptotic behavior of the amplitude and cancel in the formation of X.

The dynamical singularities of X come from three sources. First, there are the poles of $B^{(3)}$. Second, there are the poles from the zeros of $\det B^{(1)}$ where the inverse of $B^{(1)}$ does not exist. Since $H^{(1)}(R)$ has poles from the zeros of $D(E,L)$ which occur at the same position in every matrix element, the matrix M will have them also. In constructing the Fredholm series for the inverse of $B^{(1)}$ the matrix M will be iterated an arbitrarily large number of times. One could therefore have the possibility that $[B^{(1)}]^{-1}$, and hence X will possess isolated essential singularities at these points although a detailed study of the convergence of these series would be necessary to rigorously establish their existence. However, the position of these singularities depends on R_0 which is arbitrary. The poles of $H^{(1)}(R)$ can be excluded from any given bounded region of the L plane by taking R_0 sufficiently large, since the norm of the kernel in Eq. (3.17) can thus be made less than 1. Singularities which move with R_0 cannot determine the asymptotic behavior of the amplitude so that for some large value of R_0 the poles of $H^{(1)}(R)$ can all be assumed to be in the left-half plane. Indeed, one can show that the kernel of Eq. (3.17) is bounded in the right-half plane so that for the truncated problem this is true. In the following we will assume such a value of R_0 has been taken.

The important point is that all these singularities of X are isolated. Since they are all dynamical, we will presume that they are all confined in a region $\mathrm{Re}L<L_0$ for some L_0. Combining the information about $Z^{(1,3)}$ and X, we may summarize the properties of $\hat{S}$ by saying that it possesses only dynamical, isolated singularities.

We emphasize that the above discussion does not exhaust the attention which must be given to the analytic properties of these amplitudes. An important point, for instance, is to show that the kinematic singularities of J cancel when X is formed. This is the case for the analogous singularities for some problems involving two particles with spin.[6] When an infinite number of l values are coupled, there is the possibility that these are essential singularities which do not straightforwardly cancel.

Some progress can be made towards justifying the above method of proof without the artifice of truncation. Instead of trying to examine the limits as the cutoffs tend to infinity, it is more fruitful to deal directly with a partial differential equation. We introduce

$$\hat{\phi}_{lp\eta}(R,\mathbf{r})=\sum_{l'p'\eta'}\hat{\phi}_{l'p'\eta',lp\eta}(R,r) \\ \times\psi_{l'}(p',r)Y_{l'}{}^{\eta'}(\beta,\alpha). \quad (4.18)$$

$$\left[\frac{\nabla_r^2}{2m}+\frac{1}{2m'}\left(\frac{\partial^2}{\partial R^2}-\frac{\hat{\Lambda}^L}{R^2}\right)+E-V(R,r,\beta)\right] \\ \times\hat{\phi}_{lp\eta}(R,\mathbf{r})=0. \quad (4.19)$$

Here, $\hat{\Lambda}^L$ is the differential operator given by

$$\hat{\Lambda}^L=L(L+1)+\mathbf{l}^2-2l_0^2-2l_+(L+l_0)-2l_-(L-l_0), \quad (4.20)$$

where the $\mathbf{l}$ are to be interpreted as the usual differential operators in β and α. This equation is not the same as the partial-wave Schrödinger equation. Indeed, that equation was an infinite set of coupled equations in η while this is a single equation.

The integral equation (3.17) and the power series (3.4) can be transcribed into this representation. Making

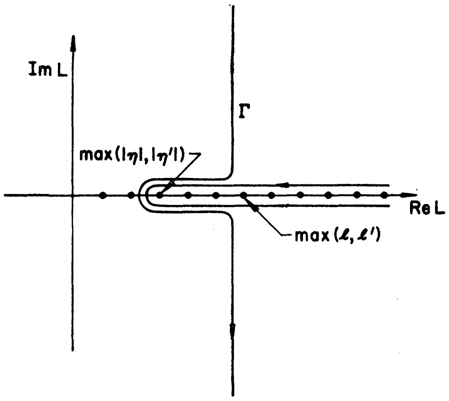

FIG. 3. Contours for making the Watson-Sommerfeld transformation.

some simple assumptions on the analytic properties of the free Green's functions, we have been able to discuss rigorously the convergence of the Fredholm series for the solutions $H^{(1,3)}(R)$ and of the power series for $J(R)$ in a region which deletes its kinematic singularities. The chief remaining problem is the construction of the matrix X which we have been unable to do rigorously. Specifically, it remains to show a boundedness condition on the matrix M to establish the convergence of the Fredholm series and to justify the cancellation of the kinematic singularities from $J(R)$.

Using the analytic properties of the amplitude outlined above, the partial-wave expansion can be written as an integral over the contour C (see Fig. 3)

$$\langle (l'p'\eta')\mathbf{P}'|T|\mathbf{P}(lp\eta)\rangle =\frac{1}{8\pi i}\int_C \frac{dL(2L+1)}{\sin\pi L}\hat{D}_{-\eta'\eta}{}^L(\pi+\varphi,\pi-\theta,\varphi) \times\hat{T}_{l'p'\eta',lp\eta}{}^L(E)\,, \quad (4.21)$$

where

$$\hat{D}_{\eta'\eta}{}^L=\rho_{\eta'}D_{\eta'\eta}{}^L\rho_\eta{}^{-1}. \quad (4.22)$$

The $\hat{D}_{\eta\eta'}{}^L$ are entire functions of L as can be seen from I. The integrand of Eq. (4.21) is therefore a meromorphic function of L in the right-half plane.

In order to deform the contour and display the Regge pole terms explicitly, the asymptotic behavior of the amplitude for large $|L|$ must be established. We will assume that this can be estimated from the Born approximation which is appropriate here since for large $|L|$ the potential can be neglected in comparison with the centrifugal barrier.

For sufficiently large ReL, the Born approximation is given by

$$T^B=\frac{1}{2}\int_0^\infty dR\phi^{0T}(R)V(R)\phi^0(R)\,, \quad (4.23)$$

where ϕ^0 is the solution-free Schrödinger equation

$$\phi_{l'p'\eta',lp\eta}{}^0 =\delta_{p'p}\delta_{l'l}\sum_\xi U_{\eta'\xi}(\tfrac{1}{2}\pi R)^{1/2}J_{L+\xi+\frac{1}{2}}(PR)U_{\xi\eta}{}^{-1}. \quad (4.24)$$

The integral converges for Re$L\geq(l+l'-1)/2$.

$$\langle (l'p'\eta')|T^B|(lp\eta)\rangle =\tfrac{1}{2}\pi\sum_{\xi\xi'\eta_1\eta_2}U_{\eta'\xi'}\int_0^\infty dRRJ_{L+\xi'+\frac{1}{2}}(PR) \times U_{\xi'\eta_1}V_{l'p'\eta_2,lp\eta_1}U_{\eta_2\xi}{}^{-1}J_{L+\xi+\frac{1}{2}}(PR)U_{\xi\eta}{}^{-1}. \quad (4.25)$$

Now if $lp\eta$ and $l'p'\eta'$ are quantum numbers of bound states, the corresponding wave functions $\psi_l(p,r)$ and $\psi_{l'}(p',r)$ decrease exponentially for large r. If we assume interparticle Yukawa potentials, a typical integral involved in computing $V_{l'p'\eta_1,lp\eta_2}(R)$ is (for equal mass particles)

$$\int d^3r\psi_{l'}(p',r)Y_{l'}{}^{\eta_1}(\beta,\alpha)\frac{\exp\{-\mu[R^2+\frac{1}{4}r^2-Rr\cos\beta]^{1/2}\}}{[R^2+\frac{1}{4}r^2-Rr\cos\beta]^{1/2}} \times\psi_l(p,r)Y_l{}^{\eta_2}(\beta,\alpha)/r^2. \quad (4.26)$$

The integrand decreases exponentially with R for all values of r and $\cos\beta$ and it is therefore reasonable to assume the integral does likewise. Since it is also a continuous function of R, and has a singularity no worse than R^{-1} at the origin, it is plausible that it is a superposition of Yukawa potentials.

$$RV_{lp\eta,l'p'\eta'}(R)=\int m_{l'p'\eta',lp\eta}(\mu)e^{-\mu R}d\mu. \quad (4.27)$$

Indeed, all these statements can be justified in detail but we will not reproduce them here.

Knowing (4.27), we can immediately take over the results of Charap and Squires[11] for contour integrals such as Eq. (4.20). The contour C can be deformed into a curve Γ (see Fig. 3) for all Re$\theta>0$ with the integral along the large semicircle vanishing. The resulting expansion is

$$\langle (l'p'\eta')\mathbf{P}'|T|\mathbf{P}(lp\eta)\rangle =\sum_n\frac{B_{l'p'\eta',lp\eta}{}^{\alpha_n}(E)}{\sin\pi^{\alpha_n}(E)}\hat{D}_{-\eta'\eta}{}^{\alpha_n}(\pi+\varphi,\pi-\theta,\varphi) +\frac{1}{8\pi i}\int_\Gamma\frac{dL(2L+1)}{\sin\pi L}\hat{T}_{l'p'\eta',lp\eta}{}^L(E) \times\hat{D}_{-\eta'\eta}{}^L(\pi+\varphi,\pi-\theta,\varphi)\,. \quad (4.28)$$

The $B_{l'p'\eta',lp\eta}{}^{\alpha_n}$ are simply related to the residues of the poles of $\hat{T}_{l'p'\eta',lp\eta}{}^L$. The asymptotic behavior in $\cos\theta$ can now be read off the above expression. With a more stringent estimate on the asymptotic behavior of the amplitude at large L, the contour could be moved further to the left. If the assumptions regarding the cancellation of the kinematic singularities of J and the position of the poles of $H^{(1)}$ were not satisfied, the asymptotic behavior could not be determined by this method.

The generalization of the preceding discussion to the scattering of a single particle from a bound state of N

[11] J. Charap and E. Squires, Ann. Phys. (N. Y.) **21**, 8 (1963).

particles is straightforward. Such an analysis is useful in a discussion of the scattering from nuclei. Again, a formal reduction of the multiparticle equation can be made to a set of coupled two-particle equations except that the number of internal variables specifying the composite object is now much larger. The same analytic properties can be derived.

The extension to the scattering of particles with spin can also be made. The behavior of the system under rotation is now more complex but the above analysis should differ in no essential way. Wick[12] has shown how to make the partial-wave expansion appropriate to this case.

V. DISCUSSION

The partial-wave Schrödinger equation continued to complex L obtained in I does not uniquely specify the scattering solutions until appropriate boundary conditions are imposed. For the physical scattering problem at an integer value of L, there is only one way to do this. For complex L, however, there are many choices of boundary condition which reduce to the physical boundary conditions at integer values of L. Each choice of the boundary conditions will result in a different continuation of the amplitude to complex values of L. We can illustrate the situation with an example.

The three-particle kinetic energy expressed in terms of r, R, l, and Λ, the orbital angular momentum of the center of mass of particles 1 and 2, can be written in its diagonal form:

$$\mathcal{T}_{l\Lambda}{}^{L}=-\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{l(l+1)}{r^2}\right)-\frac{1}{2m'}\left(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}-\frac{\Lambda(\Lambda+1)}{R^2}\right). \tag{5.1}$$

For small values of r and R, we must impose the following boundary condition on the wave function in order that the solutions be regular

$$\begin{aligned}\psi_{l\Lambda}{}^{L}&\to R^{\Lambda}F(r)\,, \quad R\to 0\\ \psi_{l\Lambda}{}^{L}&\to r^{l}G(R)\,, \quad r\to 0\,.\end{aligned} \tag{5.2}$$

Recalling that $\mathbf{l}+\mathbf{\Lambda}=\mathbf{L}$, this can be written, with L still an integer, as

$$\begin{aligned}\psi_{l\Lambda}{}^{L}(r\,R)&\to R^{L+\xi}F(r)\\ \psi_{l\Lambda}{}^{L}(r,R)&\to r^{l}G(R)\end{aligned} \qquad |\xi|\leq\max(L,l) \tag{5.3}$$

or

$$\begin{aligned}\psi_{l\Lambda}{}^{L}(r,R)&\to R^{\Lambda}F(r)\\ \psi_{l\Lambda}{}^{L}(r,R)&\to r^{L+\zeta}G(R)\,.\end{aligned} \qquad |\zeta|\leq\max(L,\Lambda) \tag{5.4}$$

Still at integer values of L, we introduce the unphysical wave functions corresponding to $L<|\xi|\leq l$ or $L<|\zeta|\leq\Lambda$. Two ways of specifying the boundary conditions on these functions which will be smooth in L when it becomes complex are to require either Eq. (5.3) or (5.4). These alternative boundary conditions will lead to solutions of a fundamentally different character for complex L since one choice specifies some solutions irregular at small R and regular for small r, while the other does the opposite. In comparing these particular alternatives, we may put the matter another way by saying that when we continue to complex L we can make either of the internal angular momenta, Λ or l, complex at the same time.

In I it was shown that there could be only one continuation of the amplitude to complex values of the angular momentum which determined the asymptotic behavior of the full amplitude through a Watson-Sommerfeld transformation. There thus must be a unique choice of boundary condition for the scattering solution of the Schrödinger equation at complex L which yields this continuation.

The boundary condition of Eq. (5.3) is the one used in this paper. This prejudice is already evident in Eq. (2.7). Recently, Newton[13] and Drummond[14] have given another continuation of the three-particle scattering amplitude which initially and finally two of the particles are bound. Along with the total angular momentum, Newton continues an orbital angular momentum associated with an interparticle distance. This corresponds to the boundary condition of Eq. (5.4). The resulting amplitude has cuts in the L plane extending infinitely far to the right. These cuts correspond to the positions of the poles in the two-body amplitudes of particles 1 and 2 but smeared out because their two-particle energy is not conserved. The discussion of this paper could be repeated for the boundary condition of Eq. (5.4) by interchanging the roles of $\mathbf{r}$ and $\mathbf{R}$, l and Λ, and p and P. In Eq. (2.7), $\psi_l(p,r)$ would be replaced by the free solution $(PR)^{1/2}J_{\Lambda+1/2}(PR)$. $B^{(1)}$ would have a term proportional to $\delta_{P'P}$ arising from the scattering of particles 1 and 2 in which 3 does not participate. As Newton shows, the P-dependent zeros of this term give rise to cuts in $[B^{(1)}]^{-1}$. It should be noted that this mechanism will produce cuts even if the equations are truncated in the way mentioned above.

The amplitude derived from the boundary condition of Eq. (5.4) does not determine the asymptotic behavior of the full amplitude in any scattering angle since there is no right-hand-most singularity. Indeed, if the discussion of this paper can be fully justified, the continuation presented here is the unique one from which the asymptotic behavior can be found for the class of amplitudes discussed.[15] This continuation maintains the simplicity of the Regge prescription for the large

[12] G. C. Wick, Ann. Phys. (N. Y.) 18, 65 (1962).

[13] R. Newton, Nuovo Cimento 29, 400 (1963); Phys. Letters 4, 11 (1963).

[14] I. Drummond (to be published).

[15] The scattering solution for another type of process, for example, the scattering of three free particles into three free particles, could require a different boundary condition to yield a proper continuation.

momentum-transfer behavior in that it has only isolated singularities.

Mandelstam[16] has demonstrated that there are cuts in the relativistic two-particle amplitudes due to the coupling with three and higher particle intermediate states. This conclusion follows from a study of the restrictions imposed by unitarity on a particular class of relativistic diagrams. His results are not in conflict with those presented here since the diagrams he considers are not present in the nonrelativistic problem. They are a type of diagram which arise from the possibility of creating virtual pairs of particles. The simple inelasticicity of a three-particle potential scattering does not seem sufficient to complicate the large momentum-transfer behavior characteristic of poles in the angular momentum plane.

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APPENDIX A: DIAGONALIZATION OF Λ^L

Define

$$\hat{\Lambda}_{lp\eta,l'p'\eta'}{}^L=\delta_{ll'}\delta_{pp'}\hat{\Lambda}_{\eta\eta'}{}^{Ll}. \tag{A1}$$

$\hat{\Lambda}^{L,l}$ is a $(2l+1)\times(2l+1)$ matrix representing the orbital angular momentum of the center of mass of particles 1 and 2. As such, it has eigenvalues $(L+\xi)(L+\xi+1)$ with $-l\leq\xi\leq l$ for integral values of L such that $L\geq l$.

Consider the function

$$\det[(L+\xi)(L+\xi+1)I-\hat{\Lambda}^{L,l}]. \tag{A2}$$

It is clearly an entire function of L with a polynomial behavior at large $|L|$. Since it vanishes on the integers $L\geq l$, Carlson's theorem[17] requires that it vanish identically. The eigenvalues of $\hat{\Lambda}^L$ for complex L are, therefore, also given by $(L+\xi)(L+\xi+1)$ with $-l\leq\xi\leq l$.

When L is an integer greater than l, the elements of the unitary transformation $U_{\eta\xi}$ which diagonalizes $\Lambda^{L,l}$ are easily computed. Denote by λ an eigenvalue of $\Lambda^{L,l}$

$$\langle LMl\eta|\lambda lLM\rangle=\sum_{m_1m_2}\langle LMl\eta|\lambda m_1lm_2\rangle\langle\lambda m_1lm_2|\lambda lLM\rangle. \tag{A3}$$

Now

$$\langle LMl\eta|lm_1\lambda m_2\rangle=\left(\frac{2L+1}{4\pi}\right)^{1/2}\int d\Omega_{\theta\phi}\int d\Omega_{\beta\alpha}\times D_{M\eta}{}^{L*}(\varphi,\theta,0)Y_l{}^{\eta*}(\beta,\alpha)Y_l{}^{m_2}(\beta_1,\alpha_1)Y_\lambda{}^{m_1}(\theta,\varphi), \tag{A4}$$

where

$$Y_l{}^{m_2}(\beta_1,\alpha_1)=\sum_{\eta'}D_{\eta'm_2}{}^l(0,\theta,\varphi)Y_l{}^{\eta'}(\beta,\alpha)$$

via a rotation of coordinates. Performing the integration, one finds in terms of Clebsch-Gordan coefficients

$$\langle LMl\eta|\lambda m_1lm_2\rangle=(-1)^{\eta+M}\left(\frac{2\lambda+1}{2L+1}\right)^{1/2}C(\lambda lL;0\eta\eta)\times C(\lambda lL;m_1m_2M). \tag{A5}$$

The sum in Eq. (A4) may be explicitly performed and an arbitrary phase chosen to give

$$\langle LMl\eta|\lambda lLM\rangle=(-1)^{\eta}\left(\frac{2\lambda+1}{2L+1}\right)^{1/2}C(\lambda lL;0\eta\eta); \tag{A6}$$

writing $\lambda=L+\xi$, we have for integral L

$$U_{\eta\xi}=(-1)^{\eta}\left[\frac{2L+2\xi+1}{2L+1}\right]^{1/2}C(L+\xi,l,L;0\eta\eta). \tag{A7}$$

The presence of Clebsch-Gordan coefficients has an obvious significance in terms of the addition $(\mathbf{L}-\mathbf{l})+\mathbf{l}=\mathbf{L}$ and the relation $\mathbf{L}\cdot\mathbf{R}=\mathbf{l}\cdot\mathbf{R}$. The transformation which diagonalizes $\hat{\Lambda}^{L,l}$ is then $\hat{U}_{\eta\xi}=\rho_\eta{}^{-1}U_{\eta\xi}$. When L is complex or real and less than l, $\Lambda^{L,l}$ can no longer be diagonalized by a unitary transformation. Since $\Lambda^{L,l}$ is symmetric, this can, however, be effected with a complex orthogonal transformation

$$UU^T=1, \tag{A8}$$

or, equivalently, if $U(L)$ is real for real L

$$U^\dagger(L)U(L)=1, \tag{A9}$$

where

$$U^\dagger(L)=[U(L^*)]^\dagger.$$

Let us examine the following continuation of $U_{\eta\xi}$ to complex L, obtained by replacing the factorials in Wigner's expression[11] by Γ functions.

$$U_{\eta\xi}=\left[\frac{(2L+2\xi+1)(2L+\xi-l)!(l-\xi)!(l+\xi)!(L+\eta)!(L-\eta)!}{(2L+\xi+l+1)!(l+\eta)!(l-\eta)!}\right]^{1/2}\times\frac{1}{(L+\xi)!}\sum_\nu\frac{(-)^{\nu+l}}{\nu!}\frac{(L+l-\nu)!(L+\xi+\nu)!}{(l-\xi-\nu)!(L+\eta-\nu)!(L+\xi-l-\eta+\nu)!}. \tag{A10}$$

[16] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963).

[17] R. Boas, *Entire Functions* (Academic Press Inc., New York, 1954), p. 153.

Using this, form the product

$$\hat{U}_{\eta\xi}\hat{U}_{\xi\eta'}^{-1}=\rho_\eta^{-1}U_{\eta\xi}U_{\xi\eta'}^{-1}\rho_{\eta'}. \qquad \text{(A11)}$$

For Re$L>l$, this expression is analytic in L. Since $U_{\eta\xi}$ as defined by (A10) is bounded[11] for large $|L|$, it is easily checked that (A11) has at most a polynomial behavior for $|L|\to\infty$. An application of Carlson's theorem again implies both (A9) for this choice of continuation and

$$\sum_\xi \hat{U}_{\eta\xi}(L+\xi)(L+\xi+1)\hat{U}_{\xi\eta'}^{-1}=\hat{\Lambda}_{\eta\eta'}^{L,l}, \qquad \text{(A12)}$$

which shows $\hat{U}$ diagonalizes $\hat{\Lambda}^L$ for complex L also. In the sense that it diagonalizes $\hat{\Lambda}^L$ for complex L, (A10) is the unique continuation of the Clebsch-Gordan coefficients.

We can now determine the small R behavior of the potential matrix elements given by Eq. (2.10) in the representation in which $\hat{\Lambda}^L$ is diagonal. In order to compute the matrix elements, we first expand the interparticle Yukawa potentials in Legendre polynomials[18] (taking the particle masses equal for simplicity)

$$\frac{\exp\{-\mu[R^2+\frac{1}{4}r^2-Rr\cos\beta]^{1/2}\}}{[R^2+\frac{1}{4}r^2-Rr\cos\beta]^{1/2}}=\sum_{n=0}^{\infty}\nu_n\left(R,\frac{r}{2}\right)P_n(\cos\beta), \qquad \text{(A13)}$$

where

$$\nu_n(x,x')=i\pi(n+\tfrac{1}{2})(xx')^{-1/2}\times J_{n+\frac{1}{2}}(i\mu x_<)H_{n+\frac{1}{2}}^{(1)}(i\mu x_>). \qquad \text{(A14)}$$

The angular integration in Eq. (2.10) may be performed explicitly to yield as a typical term

$$V_{l'p'\eta',lp\eta}(R)=2m'\delta_{\eta\eta'}\sum_n C(lnl';\eta0\eta)C(l'nl;000)\times\int_0^\infty dr\psi_{l'}^*(p',r)\nu_n\left(R,\frac{r}{2}\right)\psi_l(p,r). \qquad \text{(A15)}$$

Since $\psi_l(p,r)\propto r^{l+1}$ for small r and $n\leq l+l'$ in Eq. (A15), the integral in Eq. (A15) for small R is proportional to R^n.

To transform $V_{lp\eta,l'p'\eta'}(R)$ to the representation where $\hat{\Lambda}^L$ is diagonal, we must evaluate

$$\sum_\eta U_{\xi'\eta}(l')C(lnl';\eta0\eta)U_{\eta\xi}^{-1}(l). \qquad \text{(A16)}$$

For integer values of L greater than max(l,l') this can be evaluated using Eq. (A7) and Eq. (6.2.1) of Ref. 2.

$$V_{l'p'\xi',lp\xi}(R)\propto\sum_n C(L+\xi,\, n,\, L+\xi',\, 000)C(l'nl;000)\times\begin{Bmatrix}L+\xi & n & L+\xi'\\ l' & L & l\end{Bmatrix}\int_0^\infty dr\psi_{l'}^*(p',r)\nu_n\left(R,\frac{r}{2}\right)\psi_l(p,r), \qquad \text{(A17)}$$

where the expression in brackets is a 6-j symbol. Equation (A17) vanishes identically if $n<|\xi-\xi'|$. Now, Eq. (A16) is analytic for Re$L>$max(l,l') and is bounded at large $|L|$. An application of Carlson's theorem guarantees that it will also vanish identically if $n<|\xi-\xi'|$. Thus, from (A17) we obtain the behavior of the potential for small R

$$V_{l'p'\xi',lp\xi}(R)\propto R^{|\xi-\xi'|}. \qquad \text{(A18)}$$

APPENDIX B: ANALYTIC PROPERTIES OF THE FREE SOLUTIONS

We may introduce solutions to the free Schrödinger equation, Eq. (3.2), by

$$\begin{aligned}h_{lp\eta,l'p'\eta'}^{(1,2)}(R)&=\delta_{ll'}\delta_{pp'}\sum_\xi \hat{U}_{\eta\xi}(\tfrac{1}{2}\pi R)^{1/2}e^{i\pi[\frac{1}{2}(\xi-L-1)-l]}\times H_{L+\xi+\frac{1}{2}}^{(1,2)}(PR)\hat{U}_{\xi\eta'}^{-1},\\ j_{lp\eta,l'p'\eta'}(R)&=\delta_{ll'}\delta_{pp'}\sum_\xi \hat{U}_{\eta\xi}(\tfrac{1}{2}\pi PR)^{1/2}e^{-i\pi[\frac{1}{2}(\xi-L)-l]}\times J_{L+\xi+\frac{1}{2}}(PR_<)\hat{U}_{\xi\eta'}^{-1},\end{aligned} \qquad \text{(B1)}$$

where $J_{L+\xi+\frac{1}{2}}$, $H_{L+\xi+\frac{1}{2}}^{(1,2)}$ are Bessel functions.[18]

The matrix δ is defined with elements

$$\begin{aligned}\delta_{lp\eta,l'p'\eta'}&=\delta_{ll'}\delta_{pp'}\sum_\xi \hat{U}_{\eta\xi}e^{i\pi(\xi-l)}\hat{U}_{\xi\eta'}^{-1}\\&=\delta_{ll'}\delta_{pp'}\delta_{-\eta\eta'}.\end{aligned} \qquad \text{(B2)}$$

The last line follows from the symmetry properties of the Clebsch-Gordan coefficients at integral values of L and an application of Carlson's theorem.[17] If l denotes the diagonal matrix with elements $l\delta_{ll'}$, then

$$j(R)=(1/2i)P[h^{(1)}(R)+h^{(2)}(R)]\delta e^{i\pi(L+l+1)}. \qquad \text{(B3)}$$

For Im$P\leq0$, $h^{(2)}(R)$ satisfies the integral equation

$$h^{(2)}(R)=P^{-1/2}e^{-iPR}\delta+\int_R^\infty dR'\frac{\sin P(R-R')}{P}\frac{\hat{\Lambda}^L}{R'^2}h^{(2)}(R'). \qquad \text{(B4)}$$

Equation (B4) can be iterated and converges uniformly[19] for all finite values of L and for $R>0$. Since $\hat{\Lambda}^L$ is an entire function of L, it follows that $h^{(2)}(R)$ is also. $h^{(1)}(R)$ obeys the same integral equation but with an inhomogeneous term, $e^{iPR}P^{-1/2}Ie^{-i\pi(L+l+1)}$, where I is the unit matrix. A similar statement can, therefore, be made about the analyticity of $h^{(1)}(R)$ but with the restriction Im$P\geq0$. By Eq. (B3), $j(R)$ is then an entire function of R if Im$P=0$. The coefficient of each term in a power-series expansion of $j(R)$ must, therefore, also be an entire function of L when Im$P=0$. However, this series converges absolutely [compare Eq. (B1) and the known expansion of the Bessel function] and $j(R)$ is thus an entire function of L for all values of P.

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[18] A. Erdelyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Chap. VII.

[19] Compare the treatment of L. Favella and M. Reineri, Nuovo Cimento **23**, 616 (1962).